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The Nature and Effect of Singularities of Plane Algebraic Curves.

By Charlotte Angas Scott.

1. In Vol. XIV of this Journal I gave an account of a geometrical method of analysing Higher Singularities, by means of which there may be found for any singularity a penultimate form involving a series of nodes with a certain number of evanescent loops. It was there stated that the method is directly applicable, in general, to the determination only of the point components of the singularity, though in certain cases it determines the inflexions. I propose now to remove this restriction, showing that the process enables us, in every case, to enumerate the *double lines* (double tangents and inflexional tangents) involved in the singularity.

The definitions usually adopted for double points and double lines, as occurring in composition, are purely algebraic, $\varkappa$ (the number of cusps) being by definition one less than the L. C. D. of the exponents in a cyclic system of expansions, and ν (the number of nodes) being defined for any singularity, simple or compound, by equating $2\nu + 3\varkappa$ to the discriminantal index; it appears at once that these definitions agree with the ordinary ones for the simple node and cusp; it is then shown that ν, $\varkappa$, being so defined for any singularity, Plücker's equations hold. The numbers τ, ι are similarly defined by means of the line equation. See H. J. S. Smith, "On the Higher Singularities of Plane Curves," 1873–6 (Proc. Lond. Math. Soc., VI, 153), and Brill, "Ueber Singularitäten ebener Curven," 1879 (Math. Ann., XVI, 348).

But an algebraic curve has a geometric existence, and the process here used for the analysis of singularities is a geometric process depending entirely on a geometric conception of singularities, even though the language used is necessarily algebraic. It seems, therefore, advisable to attempt a harmonious geometric treatment of singularities (based on Plücker's views, given in his Theorie der

Algebraischen Curven, pp. 200–207, here reproduced in substance in sections 2, 3, 6, 9, 10), depending on precise geometric definitions, making use of a geometric method of analysis, and leading to a natural classification of singularities that shall give due weight to the point and line conceptions. It will then be necessary to prove that Plücker's equations hold for singularities thus defined; this is only partially accomplished in the present paper. The various relations proved by H. J. S. Smith and Brill present themselves, expressing simple geometric facts.

2. The object of our investigation is a certain geometric form, a *curve*. We define this by means of an equation connecting the coordinates of any point on it; this amounts to considering the curve as traced by a point moving subject to the law expressed by the equation. But we may also define the curve by means of an equation connecting the coordinates of any tangent; this amounts to considering the curve as enveloped by a line moving subject to the law expressed by this equation. The curves with which we are concerned are those only for which these laws are expressed by rational integral algebraic equations. Adopting the first view, we may consider the curve as the limit of a polygon defined by its vertices; adopting the second view, the polygon is regarded as defined by its sides. We have thus the dual conception of the curve, as described by a point, enveloped by a line, "the point moving continuously along the line, the line rotating continuously about the point" (Plücker).

3. Now on an arc so considered, it follows from the definition that the line elements may be derived from the point elements by joining each to the next; and that the point elements may be derived from the line elements by marking the intersection of each with the next. Consequently *either* equation, point or line, will yield us all the properties of the arc.

But suppose it happens that either part of the characterisation becomes incorrect? It may happen that the point gives up its continuous motion without any effect on the continuous rotation of the line; the point, that is, "attains a limiting position and then moves in the contrary direction" (p. 202). This we naturally expect to have an effect on the point equation, but we cannot expect any trace of it in the line equation. Again, says Plücker, this singularity may occur in the motion of the line; we do not expect to discover this if we confine ourselves to the point equation. And in the third place, these singularities may occur simultaneously. Note then that in the first case we must not expect the tangents to be derived by joining each point to the next; and in the second case, the points cannot be derived by marking the intersection of the tangent with the next.

Plücker's view is then that at the simplest point singularity, the cusp, the tracing point travels along the tangent till it attains a certain position O, and then turns back along the tangent. Now the elementary theory of maxima and minima teaches us to regard a maximum or minimum as a stationary value or position, characterised by the coincidence of two consecutive elements (of magnitude or position). There being then a cusp at O, we may consider the tracing point as occupying the position O *twice*, so that representing its course symbolically by PO_1O_2Q, consecutive positions of the enveloping line are PO_1, O_2Q; i. e. PO, OQ (Fig. 1). Note in the first place that the "tangents at a cusp" are *consecutive*, not *coincident*, for coincident tangents cause a *line* singularity; and in the second place, that the line O_1O_2 is *not in any sense a tangent.* This may be further elucidated by comparison with the reciprocal singularity, the simple inflexion; the two points of contact of an inflexional tangent are consecutive, not coincident; and we do not regard *any* point on the inflexional tangent as a point of the curve.

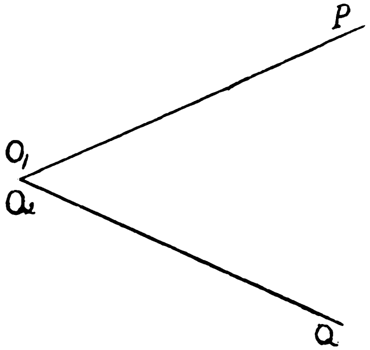

FIG. 1.

4. De Morgan (1856, Camb. Phil. Trans., IX, part 4) appears inclined to adopt the view that at a cusp dy/dx has an infinite number of coexisting values. He states this possibility with reference to the evanescent oval; but the context

suggests that he has also the evanescent loop in mind. This would agree with the view that the tangent at a cusp makes a half turn, the point suffering no singularity in its motion. Now the accepted penultimate form for a cusp, the evanescent loop (Fig. 2), appears to justify this; the tangent does turn through 180°, and the motion of the point along the tangent is unaffected. But if we accept this view for the cusp, we must accept the reciprocal view for the inflexion. In the penultimate form we have an asymptote inclined at a vanishing angle to a double tangent whose points of contact are infinitely near together (Fig. 3); thus if we account for the cusp by means of a sudden half turn of the tangent, we must account for the inflexion by means of a sudden flight through infinity, by which half a complete line is described.* Now this explanation ensures the proper zero values for $ds/d\phi$ and $d\phi/ds$ at the cusp and inflexion; but it makes the inflexion a point singularity, and the cusp a line singularity, contrary to what we know to be the case. Adopting this view really amounts to holding on to the penultimate form instead of taking the last step, which, as usual, introduces a discontinuity, shown here by the rejection of a linear factor from the reciprocal equation; this factor representing, when we form the line equation of a curve with a cusp at O, the point O, and when we form the point equation of a curve with an inflexional tangent ω, the line ω. It involves our confining ourselves to curves without cusps or inflexions, which, of course, do not exist (as proper curves) for an order greater than 2.

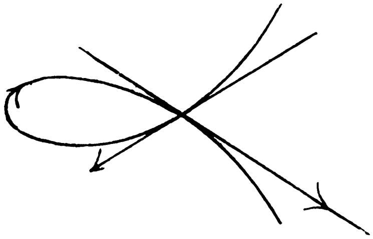

FIG. 2.

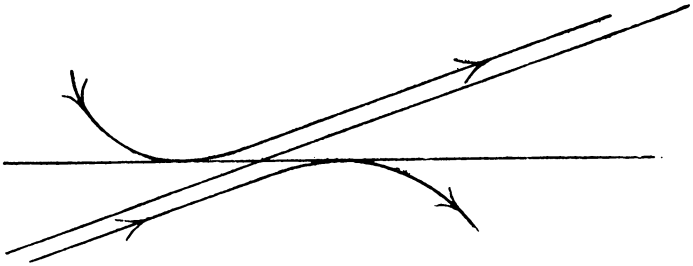

FIG. 3.

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[*] I here adopt the view reciprocal to that in which the angle about a point is considered as four right angles, and regard the complete line as consisting of the line taken twice, the upper "edge" continuous with the lower "edge," and therefore two passages through infinity being required for the complete description of the line.

As an additional example of the care needed in drawing conclusions as to the line element from the point diagram or the point equation, and vice versa, consider the tacnode formed by the simple contact of two branches, properly described as having in common one point and the tangent there, a description which is reciprocal to itself. The point equation expresses this "the branches have two consecutive points common," the line equation says "the branches have two consecutive lines common." But if we combine these and say the branches have two points and two tangents common, we are giving what is properly the description of the oscnode. Fig. 4 represents the point and line elements of a tacnode; $1 = 1'$ and $2 = 2'$ are the common points; $12 = 1'2'$ is one of the common tangents; and if we consider the line elements before 1 as *not* coincident, (therefore, by continuity, necessarily consecutive) then we must consider 23, $2'3'$ as coincident; but 3 and $3'$ must be regarded as consecutive, not as coincident points.

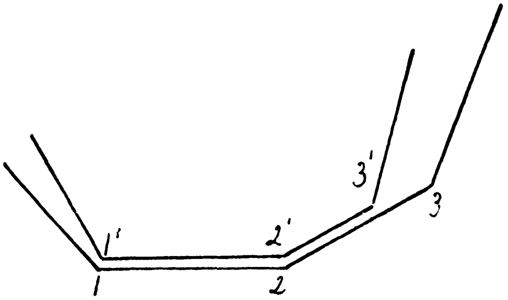

FIG. 4.

5. These examples illustrate the fact that the expression by means of points is a convention, and the expression by means of lines a different convention; and as these, or the penultimate forms derived from them, may express different parts of the truth, we cannot expect to be able to deduce the whole truth from one expression. The penultimate form in each of the diagrams used in the method of inversion is derived from the point equation, and must therefore be cautiously used in drawing conclusions as to line singularities; we have no grounds for assuming it to be the penultimate line form, and in fact the one diagram for the resolution of the simple cusp shows that this is not the case.

Further, the single algebraic equation is deceptive in another way, in that it does not discriminate between *consecutive* and *coincident*. Thus e. g. the point equation for the tangents at a cusp gives coincident lines, whereas we have seen that they are really consecutive; the tangents at a tacnode or at a cusp of the second species are properly coincident.

6. When we consider a curve, then, as defined by its point equation, cusps and nodes present themselves as singularities; when by its line equation, inflexional and double tangents are the singularities; when however we use the intrinsic equation, cusps and inflexions are the singularities. Plücker regards nodes and double tangents as accidents of position, and not real *singularities*; stating that he confines himself to singularities on a single arc. But we shall find that the occurrence of more than one proper singularity at a point, even on a single arc, involves the occurrence of nodes and double tangents; e. g. two cusps cannot exist without a node; a cusp and inflexion induce at least one node and one double tangent. We shall therefore frame our definition of singularities so as to include these. Plücker defines a singularity as caused by the turning back of the element; if this occurs twice, three times, etc., we have at the point two, three, cusps or inflexions. For purely geometric purposes it is perhaps more convenient to express this in terms of successive elements, instead of in terms of a single moving element; and this has the further convenience of enabling us to include nodes and double tangents.

It is occasionally convenient to adopt a symbolic notation; we use letters for positions in the plane, and when the describing element occupies a position more than once, we use suffixes, consecutive or not, as the case may be. Thus the symbol for a cusp is PO_1O_2Q, or $PO_{12}Q$; for an inflexion, po_1o_2q or $po_{12}q$; for a loop, $PO_1Q\ldots.RO_{n+2}S$; and for an evanescent loop, $\mathop{L}\limits_{n=0} PO_1Q\ldots.RO_{n+2}S$.

7. We therefore formulate our conception of the singularities of a curve as follows: Any arc of the curve may be considered as yielding a succession of point elements occupying certain of the point positions in the plane; and a succession of line elements, occupying certain of the line positions in the plane; the coincidence of two point elements gives a double point, which is a cusp if the elements are consecutive, otherwise a node; the coincidence of two line elements gives a double line, which is an inflexional tangent if the elements are consecutive,

otherwise a double tangent. If k point elements coincide at O, the lowest terms in the point equation are of degree k, and the singularity is said to be of order k; similarly if l line elements coincide, the lowest terms in the line equation are of degree l, and the singularity is said to be of class l. Using a rather elliptical expression, we may say that the singularity has a point equation of degree k, and a line equation of degree l. Suppose now we have at O k consecutive elements (symbolically, the point is $PO_1O_2 \dots O_kQ$); we have then $k(k-1)/2$ coincidences, of which $k-1$ $(O_1-O_2,\ O_2-O_3,\ O_3-O_4, \dots O_{k-1}-O_k)$ are coincidences of consecutive elements; the number of cusps in this singularity is therefore, by definition, $= k-1$; and similarly we may have a singularity of class l in which there are $l(l-1)/2$ double lines, of which $l-1$ are inflexional tangents.

8. Thus the special characteristic of the cusp and inflexional tangent, which differentiates them from the node and double tangent, is the *consecutiveness* of the elements involved in the coincidence,* and it would be convenient to have a special term to express this. Now it is precisely this property of consecutiveness in the point elements that is signalised by the presence of a branch point when ordinary analytical geometry is regarded from the standpoint of the theory of functions. But there is a decided practical inconvenience in borrowing a term from one subject to use in another, for its region of applicability is apt to be different in the two subjects; and further, in this case, there is the strong objection to the term branch point—that it is not available for the line theory. We need in analytical geometry words to express the consecutiveness of point or line elements, these being (1) not coincident, as e. g. the consecutive points in which a proper tangent meets the curve; (2) coincident, as e. g. these points when the line is not a tangent, but a line through a cusp. *Conjunction* and *consecution*, though hardly expressive enough, might serve. Thus for a curve of class n with κ cusps, there are $n+\kappa$ point conjunctions with reference to any point, of which n are relative, κ absolute; or we may say there are κ consecutions; and for a curve of order m with ι inflexions, there are $m+\iota$ line

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* It is foreign to the design of this paper to discuss the purely algebraic treatment of the question; but to facilitate comparison it may be mentioned that it is easy to show from the expansions that a point describing a superlinear branch *in conformity with the algebraic law expressed by the integral equation* must pass from expansion to expansion at the superlinearity; i. e. the consecutive point is to be found on a different expansion.

conjunctions with reference to any line, of which m are relative, ι absolute; or we may say there are ι consecutions. And again, the simplest superlinearity of order (or class) 3 contains 3 point (or line) coincidences, of which 2 are consecutions.

9. Plücker then points out that his definitions give at once the ordinary values for the radius of curvature; the same holds good for the modified definitions here adopted. The curve is defined in terms of s and ϕ, and the radius of curvature is $ds/d\phi$. Note in passing that it is better not to regard s as a function of ϕ, for in so doing we should be led to make the tacit assumption that ϕ, our independent variable, increases continuously, which would hamper us in the treatment of line singularities.* We therefore regard s and ϕ as functions of an independent variable t, and we have $\rho = \mathop{L}\frac{ds}{dt} \Big/ \frac{d\phi}{dt}$. Thus at a single cusp, by definition, $ds/dt = 0$, $\therefore\ \rho = 0$; and at an inflexion, $d\phi/dt = 0$, and $\therefore\ \rho = \infty$; and similarly if there are κ cusps, ι inflexions, so that $\kappa\,(= k - 1)$ and $\iota\,(= l - 1)$ differential coefficients vanish, we have

$$\rho = \mathop{L}\frac{d^k s}{dt^k}(\delta t)^k \Big/ \frac{d^l \phi}{dt^l}(\delta t)^l$$

Thus if $k < l$, $\rho = \infty$;
if $k > l$, $\rho = 0$;
and if $k = l$, ρ is determinable.

10. In referring to the geometrical representation, it will be convenient occasionally to use the term *path* for the (real) path of the tracing point to or from O; so that an ordinary linear branch is made up of two paths joining at O; and a superlinear branch is made up of two paths, with some sort of junction at O. These two paths, with whatever the point does at O, make the *arc*, this word being used to denote the complete branch, linear or superlinear. The appearance of an arc depends on the way the two paths PO, $O'Q$ which meet at O,

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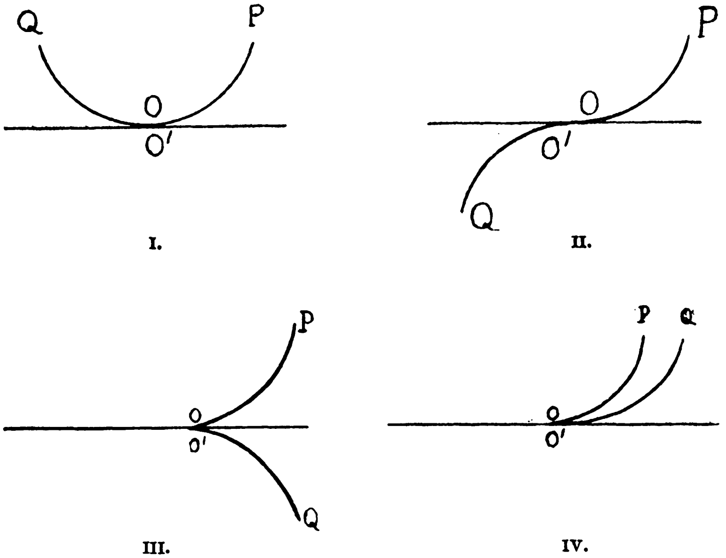
* The same thing may be noticed in the use of the ordinary Cartesian equation, where, regarding y as a function of x, we tacitly assume that x increases continuously. This is all right in the purely algebraic theory, but it is an interesting question how far the imaginary "bridges" from circuit to circuit so introduced really belong to the geometric curve. They have not the organic connexion with the curve that the imaginary branches at a singularity have.

and have necessarily the same or consecutive tangents, are placed, and not at all on the way the transition from O to O' is accomplished. There are therefore four possible varieties, as represented in Fig. 5.



I. II. III. IV.

Fig. 5.

Now these depend only on k and l (Plücker, p. 207); for an even number of cusps (i. e. odd k) has no ultimate effect on the progress of the point; thus for I, II, k is odd; for III, IV, k is even; and an even number of inflexions (i. e. odd l) has no ultimate effect on the rotation of the line; thus for I, III, l is odd; and for II, IV, l is even.

But these arcs may be combined in any way, so producing composite singularities. From our definitions it follows that in a composite singularity the number of cusps and the number of inflexions are respectively the sums of those afforded by the separate superlinearities; but that the number of nodes and of double tangents will be augmented by the intersections and contacts of the complete branches. We shall speak in general of the singularity as containing δ double points (dps), of which ν are nodes, $\varkappa$ cusps; and λ double lines (dls), of which τ are double tangents and ι inflexional tangents.

11. The ordinary proofs show (or as in §7 it may be shown) that a singularity of order q and class r contains certainly $q(q-1)/2$ dps, and $r(r-1)/2$ dls.

But the simple example of a tacnode shows that the q-pt indicated by the equation may be but one of a series of consecutive multiple points; we shall therefore at present speak of a q-element, meaning one whose equation is of degree q, and which therefore contains certainly a q-point (line) and possibly other multiple points (lines) not yet apparent; say it contains a certain number of *latent* multiple points (lines). The main problem in the analysis of any singularity is to determine the latent double elements.

A process for determining the double points was explained in the article already referred to, Am. J., XIV, p. 301. Applied to a q-element, it clears away the obvious $q(q-1)/2$ coincidences, at the same time showing which of these are coincidences of consecutive points; for it represents the q points that come together at O by q points on the base line; and as the consecutiveness of points is preserved in the transformation, we have only to consider which of these q points are consecutive; thus e. g. $y^3 = x^4$ gives $y^3 = x$; the three points on the base are consecutive, we have therefore for the original singularity the symbol $PO_{123}Q$, and there are two cusps. But multiple points adjacent to O are unaffected by the transformation; thus the transformed equation presents to our view a certain number of q'-points—or rather elements—where $\Sigma q' \leqq q$, and these q'-elements must be further resolved. A superlinear branch in every case leads to a single q'-element, where $q' \leqq q$; and proceeding in this way, the singularity is gradually untwisted. Since $q' \not> q$, we see that the q-element certainly exhibits at once by its equation its highest component. Thus the q-element contains a q-point with certain latent multiple points, no one of order $> q$. If these latent multiple points contain Q dps, we have therefore

$$\delta = q(q-1)/2 + Q.$$

Further, it was shown that for a single superlinearity of order k,

$$\varkappa = k - 1.$$

Regarding the reciprocal equation of degree r in the same way, we have

$$\lambda = r(r-1)/2 + R,$$

and for a single superlinearity of class l,

$$\iota = l - 1.$$

We have now to show how this same method of quadric inversion may be applied to determine λ and ι, and hence τ, without forming the reciprocal equation. The expressions just given for δ and λ are for any singularity, whether composite or simple. We first show how q and r present themselves; we then prove that in every case $R = Q$.

12. It has already been pointed out that any singularity is caused by the superposition of a number of superlinearities. Now if these come at the same point, with different tangents, no latent dps and no latent dls are caused; and similarly for the same tangent with different points of contact. Thus for our present purpose we have only to consider a q-element with all the tangents coincident.

A superlinearity at O, tangent OZ, when inverted gives rise to a superlinearity at Z; let OZ, the *invertor*, meet this in l points, and let the base meet it in k points, so that k is the order of the singularity O. The tracing point P (describing Z) (Fig. 6a) has therefore to cut the base k times and the invertor l times. Now in order that it may cut the base k times, it must oscillate along OZ, making $k - 1$ turnings back (Fig. 6b); any such turning back is represented by consecutive points (which may be also coincident) on the base, and thus it produces a coincidence of consecutive points at O, i. e. a cusp; thus

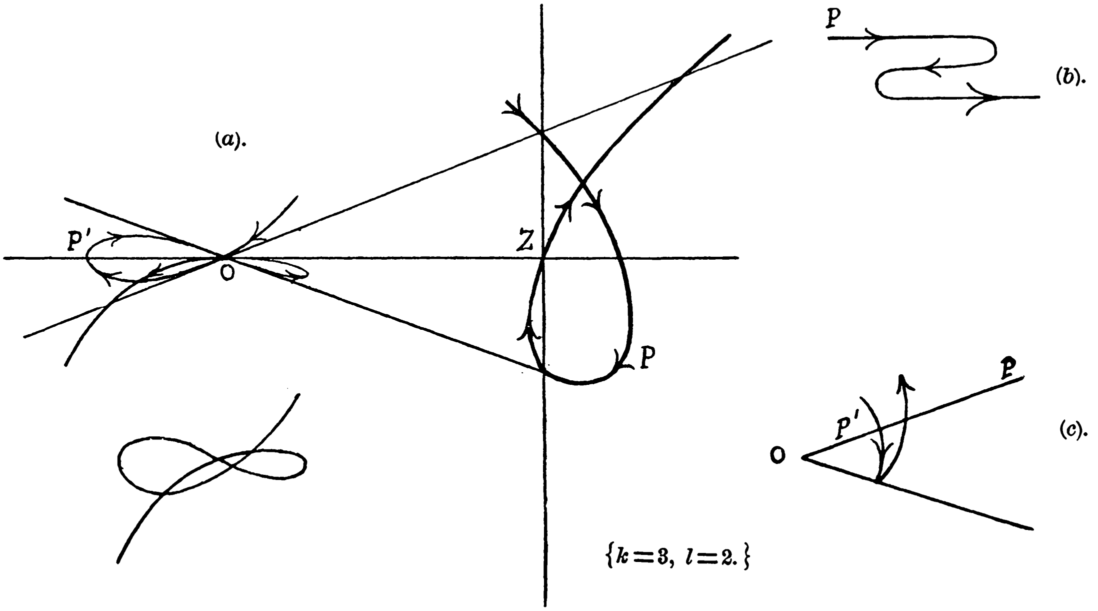


Fig. 6.

the number of cusps is $k-1$, and k is the order of the singularity O, which agrees with what we have already said as to the degree of the point equation.

Again, in order that P may cut the invertor l times, the line OP, whose motion represents that of the tangent at O (viz. OP'), must take up the position OZ l times, and consequently attain a limiting position and turn back $l-1$ times (Fig. 6*c*). Now it is only at these turnings back that two consecutive positions of the tangent OP' can coincide, i. e. these $l-1$ are the only inflexional tangents that can present themselves. Thus OZ presents itself as an l-line, giving $l(l-1)/2$ dls, of which $l-1$ are inflexional tangents; the class of the original singularity is therefore l, and λ is at least $l(l-1)/2$. Thus the order and class of the superlinearity at O are given by the number of points in which the first inverse is met by the base and the invertor.

Now consider the singularity made by the superposition of j superlinearities with the same tangent OZ at O. Let these be of order and class k_1l_1, k_2l_2, etc. Then the first inverses, all at Z, are met by the base and invertor in Σk and Σl points. We have already defined the order of the singularity as Σk, the degree of the original point equation; and similarly the class is Σl; thus the order and class are given as before by the number of points at Z on the base and invertor. Moreover, since $\varkappa_1 = k_1 - 1$, $\iota_1 = l_1 - 1$, we have

$$\varkappa = \Sigma(k_1 - 1) = k - j,$$
$$\iota = \Sigma(l_1 - 1) = l - j.$$

Note then that OZ meeting the inverse in l points, and the base meeting it in k points, the number of points in which OZ meets the original singularity is $k+l$. Thus the order of the singularity at O is given by the degree, k, of the lowest terms in the point equation, and subtracting this from the number of points in which the tangent meets the singularity, we have the class, l.

For our singularity we have now as minimum values for δ and λ, $k(k-1)/2$ and $l(l-1)/2$. Let the first inverse contain, in all, h dps, these being determined as already explained, so that $\delta = k(k-1)/2 + h$; write similarly $\lambda = l(l-1)/2 + h'$; we have now to show that in every case $h' = h$.

13. The determination of the number of dls really depends on our knowledge of the general point nature of a singularity. We have a k-pt, from which proceed a certain number of strings of multiple points, in general of decreasing

order. Thus the singularity illustrated in Figs. 26, 27 (loc. cit.) gives a sextic point; the dps 4, 5, 6 indicate three series of dps; in two of these we have only the one member, 5, 6, but in the remaining one we have a series of four consecutive dps. Some of these series may be closed altogether or in part, the branches uniting to form a cusp; and finally we have emerging $2j$ branches, 2 for each superlinearity in the composite singularity. If now N is a k'-pt consecutive to O, k' of the k tangents at O pass through N; in the example already chosen the six tangents at O are coincident in pairs, leading to 4, 5 and 6; we have thus 3 dls. But also there are the emerging free arcs, which have contact of a certain order, and therefore yield a certain number of dls. So we have to take into account the dls that lead from O to the consecutive multiple points, those from the first rank of latent multiple points to the second, and so on.

Now two free branches that have v consecutive points common have in general v double tangents; but if the branches have inflexions at O, this number is increased. Let the two branches, each of order 1, be of class l_1, l_2. If we have two superlinearities of orders k_1, k_2, each of class 1, these lead on inversion to linear branches having k_1 and k_2 points on the base; thus the number of dps in the inverse is the smaller of the two numbers k_1, k_2; let this be k_1; the total number of dps is therefore $(k_1+k_2)(k_1+k_2-1)/2+k_1$, of which k_1-1+k_2-1 are cusps. Reciprocally, for the number of dls in the case to be considered, if $l_1 \ngtr l_2$, we have

$$\lambda = (l_1+l_2)(l_1+l_2-1)/2+l_1.$$

Now the singularity formed by the two branches is of order 2, class l $(=l_1+l_2)$; and if $l_1 < l_2$, l_1 is the number of latent dps, for the two branches have respectively l_1+1 and l_2+1 points on the common tangent; thus

$$\lambda = l(l-1)/2+h;$$

but if $l_1 = l_2$, there are other latent dps, say v_1 in one direction from O, v_2 in the other; but these, indicating a certain degree of contact between the curved branches, give respectively v_1 and v_2 dts, so that we have

$$\lambda = l(l-1)/2+l+v_1+v_2,$$

i. e. as before

$$\lambda = l(l-1)/2+h.$$

If now we have any number of such branches, since the expression for λ brings in the dls for the two branches separately, this part must not be repeated in adding up the numbers found for $\lambda_{12}\,\lambda_{13}\,\lambda_{23}$, etc. We have

$$\begin{aligned} 2\lambda_{12} &= (l_1+l_2)(l_1+l_2-1)+2h_{12} \\ &= l_1(l_1-1)+l_2(l_2-1)+2l_1l_2+2h_{12}, \\ \therefore\ 2\Sigma\lambda_{12} &= \Sigma l_1(l_1-1)+2\Sigma l_1l_2+2\Sigma h_{12} \\ &= \Sigma l_1^2+2\Sigma l_1l_2-\Sigma l_1+2\Sigma h_{12} \\ &= \{\Sigma l_1\}^2-\Sigma l_1+2\Sigma h_{12} \\ &= l^2-l+2h, \end{aligned}$$

and so we obtain finally $\lambda = l(l-1)/2+h$.

Now the diagram for the penultimate form of a j-fold singularity of order k contains $k-j$ evanescent loops; if we remove the tips of f of these loops, so making the corresponding cusps into nodes, we obtain a diagram for the penultimate form of a $(j+f)$-fold singularity containing $k-(j+f)$ cusps; thus k and δ, and therefore h, are unaltered, but l and ι, and therefore λ, are in general altered. It is easiest to trace the changes when we start with the free branches, and restore the tips, i. e. reintroduce the cusps, one at a time. We have then to show that if $\lambda = l(l-1)/2+h$,

$$\lambda' = l'(l'-1)/2+h;$$

i. e. we have to show

$$\lambda'-\lambda = \tfrac{1}{2}\{l'(l'-1)-l(l-1)\}.$$

Now we shall find that in every case

$$\begin{aligned} &l'-l=1 \text{ or } 0\text{; if then} \\ &l'-l=0\text{, we have to show } \lambda'-\lambda=0 \\ \text{and if }\ &l'-l=1\text{, we have to show } \lambda'-\lambda=l. \end{aligned}$$

The tangents at a cusp are (i) consecutive, (ii) coincident; if consecutive, the arcs in the vanishing angle turn their concavities towards one another; if coincident, the branches are on (a) the same, (b) opposite sides. Further, (iii) removing a cusp may remove intersections with other branches, so this must be considered in restoring the cusp. (Fig. 7.)

In (i) l and λ are unaltered;

in (ii, a), l is unaltered; a double tangent is changed into an inflexional tangent, but this leaves λ unaltered;

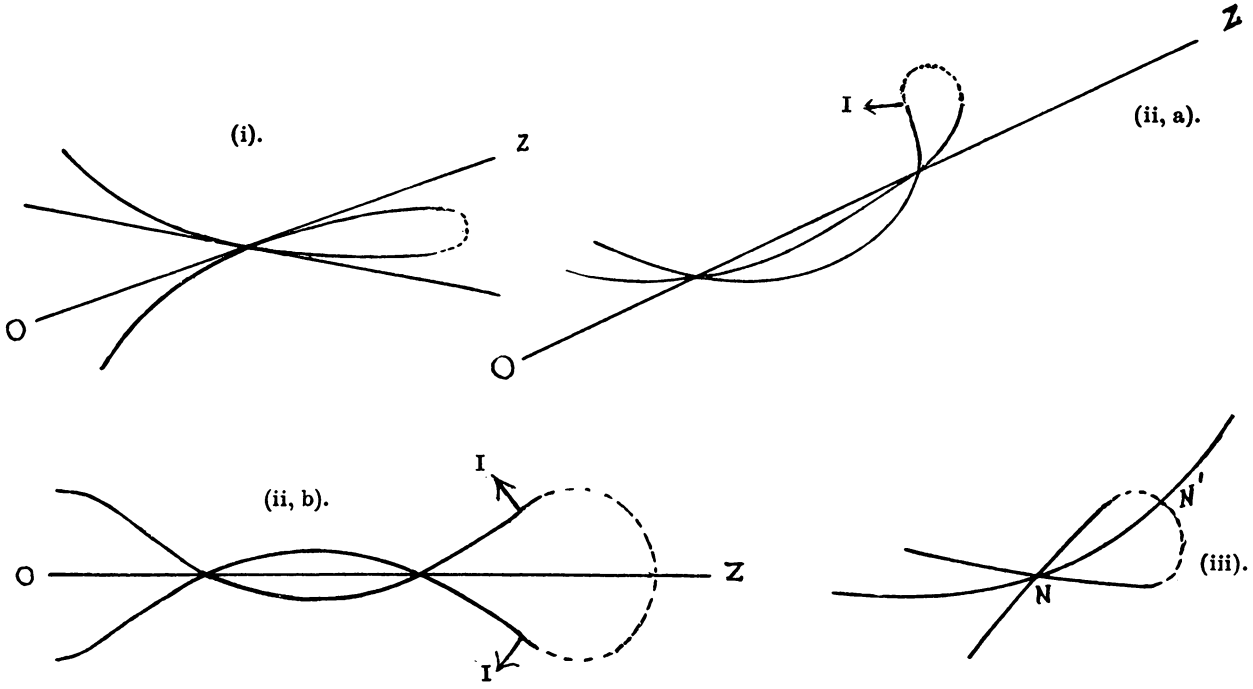


FIG. 7.

in (ii, b), l is increased by 1, ι by 2, without any corresponding diminution in τ, but with an actual increase. For OZ, which was an l-line, becomes an $(l+1)$-line; thus λ is increased in all by $\frac{1}{2}\{(l+1)\,l - l\,(l-1)\}$ i. e. by l;

in (iii), the joining of the free ends introduces one more dp N'; it does not additionally affect l, ι, but as NN' is now a dt, λ is increased by unity, corresponding to the increase of h by unity.

Thus throughout the changes we have

$$\lambda = l\,(l-1)/2 + h\,;$$

i. e. in any singularity with all the tangents coincident, *the number of latent dps is equal to the number of latent dls*; and as the superposition of singularities with different tangents introduces no latent double elements, this relation is universally true. We have then for a j-fold singularity the equations

$$\delta = k\,(k-1)/2 + h\,; \quad \lambda = l\,(l-1)/2 + h\,;$$
$$\varkappa = k - j\,; \qquad \iota = l - j\,.$$

14. Examples of the four kinds of superlinearity, enumerated in §10, with the values of (k, l, h). (Fig. 8.)

I. $(y - x^2)^3 = x^2y^3$; $(3, 3, 4)$; $\therefore \varkappa = 2, \nu = 5$; $\iota = 2, \tau = 5$.
II. $(y - x^3)^3 = x^2y^3$; $(3, 6, 7)$; $\therefore \varkappa = 2, \nu = 8$; $\iota = 5, \tau = 17$.
III. $(y^2 - x^3)^3 = x^4y^4$; $(6, 3, 7)$; $\therefore \varkappa = 5, \nu = 17$; $\iota = 2, \tau = 8$.
IV. $(y^2 - x^3)^2 = x^7 + 4x^5y$; $(4, 2, 2)$; $\therefore \varkappa = 3, \nu = 5$; $\iota = 1, \tau = 2$.

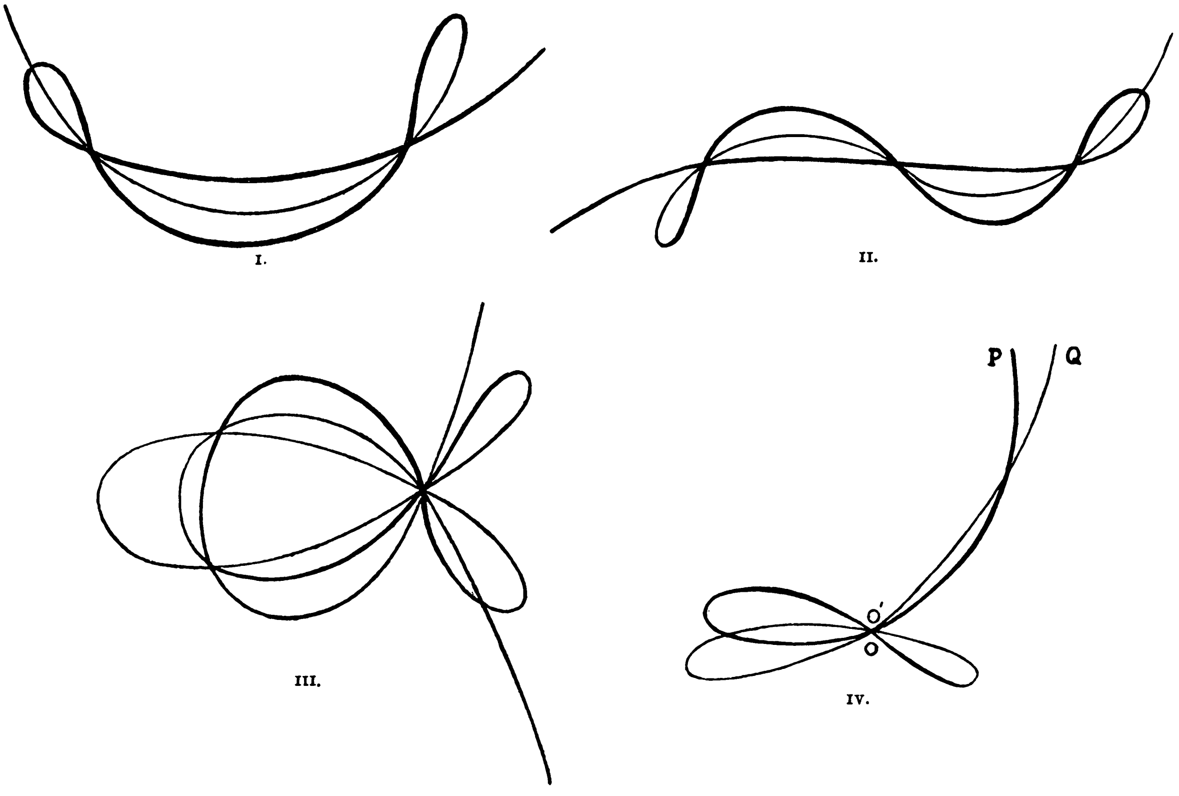


FIG. 8.

The example chosen as an illustration of case IV is one of those given by H. J. S. Smith (loc. cit. p. 168). He regards the superlinearity as composite, (k and l being even), and states that the two real branches

$$y = x^{\frac{3}{2}} + x^{\frac{7}{4}}, \ (PO) \text{ and}$$
$$y = x^{\frac{3}{2}} - x^{\frac{7}{4}}, \ (QO')$$

"appertain to a real ramphoid cusp." But as we are dealing with algebraic curves, these two branches have no existence without their conjugates; geometrically, these are needed for the transition from O to O'. We therefore cannot regard the superlinearity as composite.

15. This quantity h, which has numerical analogies with the deficiency of a curve, may be conveniently called the *excess* of the singularity; and the relation proved in §13 shows that for any singularity the point excess and the line excess are equal. We have then a dualistic organic classification of superlinearities, by order, class, and excess, $(k, l; h)$ corresponding to the $(m, n; p)$ classification of curves; and just as in the case of curves, these characteristic numbers do not lead to a unique determination; for this, in the case of even a single superlinearity, we require to know the series of multiple points and their arrangement. The Plückerian characteristics are expressed in terms of $k\ l\ h$ by the equations given in §13, from which we obtain

$$\nu = (k-1)(k-2)/2 + h, \quad \tau = (l-1)(l-2)/2 + h;$$

and we can derive a series of equations analogous to those given in Salmon's Higher Plane Curves, p. 65; e. g.

$$k^2 - 2\nu - 3\varkappa = l^2 - 2\tau - 3\iota;$$

$$2(\tau - \nu) = (l-k)(l+k-3)$$

$$= (\iota - \varkappa)(\iota + \varkappa - 1) \text{ (cf. with H. J. S. Smith, p. 166);}$$

$$\nu - (k-1)(k-2)/2 = \tau - (l-1)(l-2)/2 = h.$$

Further, as to the analogy of the excess h with the deficiency p; when the singularity is resolved into a series of multiple points, the maximum number of such multiple points is $h + 1$; for the maximum is plainly attained if all the latent multiple points are dps, and then there must be h of them, so that counting in the central k-pt we have $h + 1$ multiple points. And again, we can determine a minimum value for h. Let the smaller of the two numbers $k, l = g$; if $g > 1$, the point Z on the first inverse is a multiple point of order g, and therefore contains $g(g-1)/2$ dps; thus $h \nless g(g-1)/2$.

16. In certain cases when k and l are given, h is known. For we start with a singularity O, of order k and class l, or say a point $(k, l + k)$. Let the quotients and remainders in the process of finding the G. C. M. of k, $l + k$ be $f, f_1, f_2 \ldots . k_1, k_2 \ldots .$ so that

$$l + k = fk + k_1, \quad k = f_1 k_1 + k_2, \text{ etc.}$$

Then f inversions along OZ lead to a point (k_1, k); f_1 inversions along the new tangent (which was the base) lead to a point (k_2, k_1), and so on. (Fig. 9.) This continues until we come to a point for which $k_r = k_{r+1}$, and then the tangent is neither of the lines with which we are dealing, so that we do not know in how many points the tangent meets the singularity, and therefore cannot say à priori how the series of multiple points continues. But if k is prime to l, the last remainder is unity, and so the series of multiple points terminates. We have then f k-pts, f_1 k_1-pts, ... f_ν k_ν-pts; consequently we have

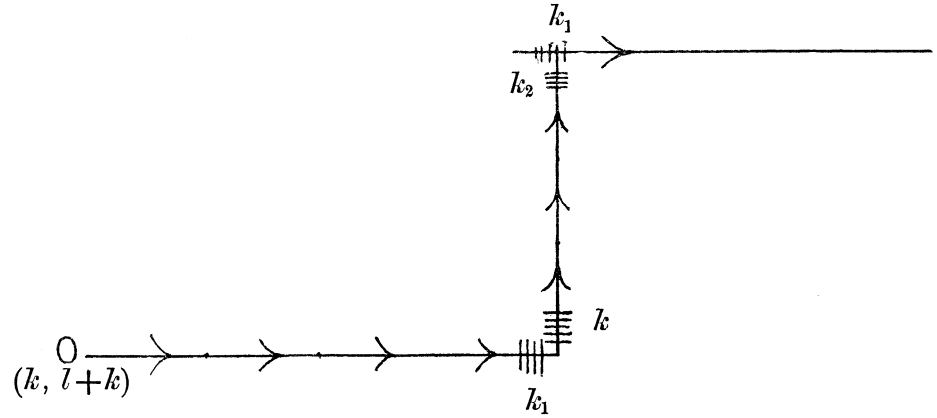


FIG. 9.

$$\begin{aligned}2\delta &= f \times k(k-1) + f_1 \times k_1(k_1-1) + \ldots. + f_\nu \times k_\nu(k_\nu - 1)\\ &= (l+k-k_1)(k-1) + (k-k_2)(k_1-1) + (k_1-k_3)(k_2-1) + \ldots\\ &\qquad\qquad \ldots + (k_{\nu-1}-1)(k_\nu-1)\\ &= (l+k)(k-1) - kk_1 + kk_1 - k_1k_2 + k_1k_2 \ldots. + k_{\nu-1}k_\nu\\ &\qquad + k_1 - k + k_2 - k_1 + \ldots. + k_\nu - k_{\nu-1} - k_\nu + 1\\ &= (l+k)(k-1) - k + 1\\ &= (k-1)(l+k-1);\end{aligned}$$

now $2\delta = k(k-1) + 2h$,

$$\begin{aligned}\therefore\quad 2h &= (k-1)(l+k-1) - k(k-1)\\ &= (k-1)(l-1);\end{aligned}$$

i. e. if l is prime to k, $h = (k-1)(l-1)/2$. (Cf. w. Brill, loc. cit., p. 355.)

17. The considerations already presented as to the arrangement of the constituent multiple points in a superlinearity show that there is a twisting

backwards and forwards on a certain fundamental curve, with, it may be, at O a "rosette" as in the point referred to in §13. In considering possible arrangements, it should be noted that if in a series of multiple points the order drops by unity, we have got to the end of that series. For the point before this one being $(k', l' + k')$, certainly k' is not unity, or we should have got to the end; now the next point is (k', l'), and we have therefore by the hypothesis $l' = k' - 1$; i. e. we have now got to a point of order $k''(= l')$ met by its tangent in $k'' + 1 (= k')$ points; the inverse of this is then of order $k'' + 1 - k''$, i. e. 1; e. g. $y^3 = x^{11}$; here we have 3pt + 3pt + 3pt + 2pt; there are consequently no more multiple points. The converse of course does not hold.

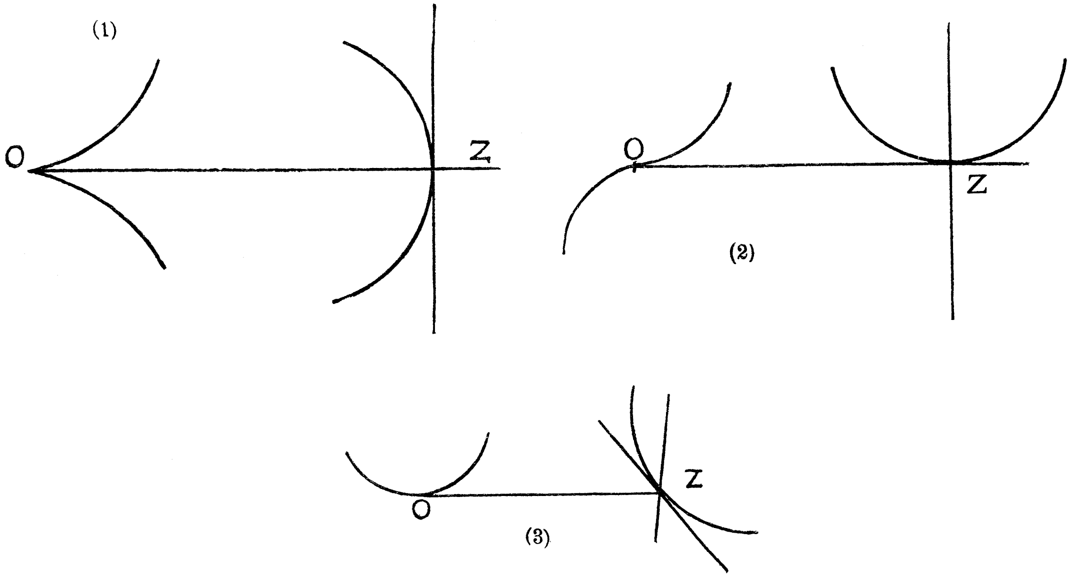


FIG. 10.

18. Since every superlinear branch can be reduced by successive inversions to an ordinary branch, we might regard any one as so many degrees removed from an ordinary arc, or say as of such a rank, counting the ordinary arc as of the first rank. Now a singularity at Z yields different results at O according as (1) the base, (2) the invertor, or (3) some other line, is the tangent at Z. (Fig. 10.) But if the arc at Z is a one-element, then inverting by process (3) gives no higher singularity at O; rejecting this, in the first rank we have one member; in the second we have *two.* (Fig. 10; (1) (2).) In the third we have

$3 \times 2 - 1$, since the inverse of (2) by the third process is to be rejected; and in general in rank g there are $\frac{1}{2}\{3^{g-1}+1\}$ members. But it hardly seems advisable to adopt this classification, for if we attempt to enumerate singularities in this way we shall be taking account of insignificant differences. Thus, e. g. $(y-x^2)^2 = x^7$ and $(y-x^2-x^3)^2 = x^7$ would be enumerated as distinct; now each has one cusp and two nodes, arranged as consecutive points on a curve of finite curvature; each has one inflexion and two double tangents; the only difference is that the foundation curve is in one $y = x^2$, and in the other, $y = x^2 + x^3$.

19. Since a composite singularity may be made by any combination of superlinearities, it hardly seems worth while attempting an organic classification. For even if we introduce j for the number of superlinearities, (§13) though the Plückerian characteristics are thus made to depend on the essential nature of the singularity, yet the four quantities k, l, h, j, do not even tell what the numbers are for the separate superlinearities; so there seems no special advantage in using them.

20. The relations proved show that two singularities related by having the same inverse, but with the base and invertor interchanged, have their l and k interchanged, and are thus reciprocal as to their symbol. Are they in reality reciprocal? We know that $(k, l; h)$ do not uniquely determine the singularity, and thus there is room for asking this question.

Take a proper conic, with $OZ\Omega$ a self-conjugate triangle. (Fig. 11.) Let QP' be the polar of P, then OP meets QP' in P', the inverse of P. Let Q be the pole of OP; then if OP is the tangent at O, and therefore coincides with OZ, Q comes at Ω, and $\Omega P'$ is the tangent to the reciprocal. Thus the singularity at Z, when inverted with ΩZ as the invertor, gives at Ω a singularity which, to a certain order of approximation, coincides with the reciprocal of O. As to the branches proceeding from the k-pt, we know that these reciprocate into branches having contact of the same order. Thus the central part of the singularities at O and Ω will be reciprocal; and the number and general arrangement of the outlying multiple (reciprocal) elements will correspond; but the two singularities will not necessarily be reciprocal.

21. It is of interest to compare the formulæ here obtained with those given by Halphen in his "Mémoire sur les points singuliers des courbes algébriques

planes" (1874) for the behavior of the evolute. The generalised evolute is the envelope of the line $O\Omega$ in Fig. 11; this is the polar of Z. For parts of the curve in the immediate neighborhood of O, with OZ as tangent, the generalised evolute and the first inverse are reciprocal polars. If we take the base conic a circle, so that IJ are the circular points, the generalised evolute coincides with the ordinary evolute for parts of the curve in the immediate neighborhood of O;

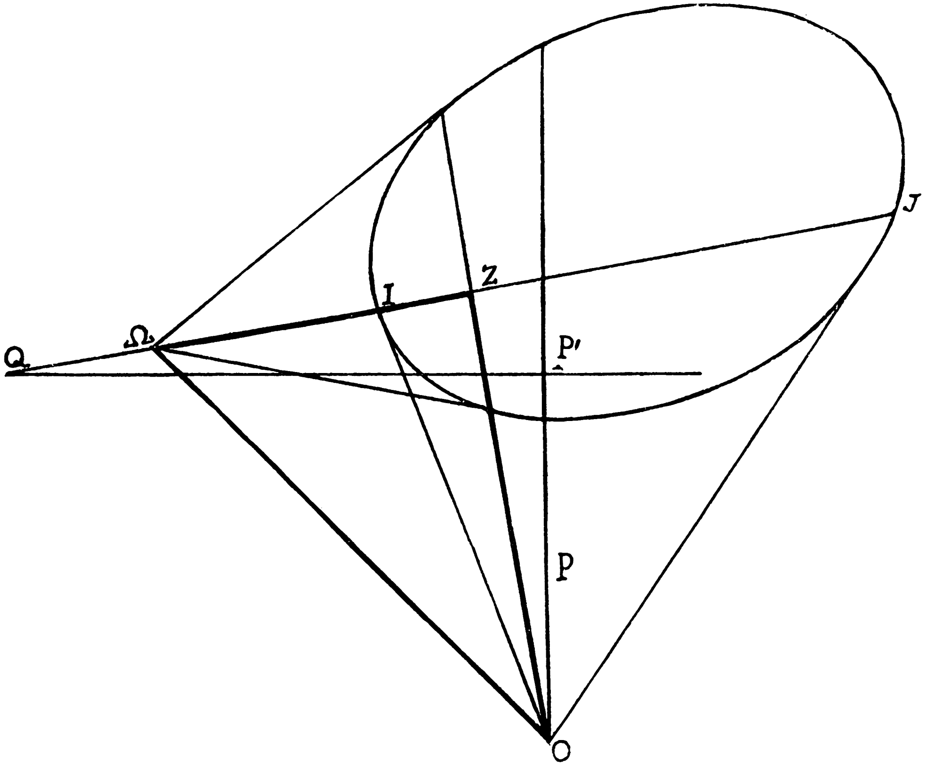


FIG. 11.

and thus for these parts of the curve the first inverse coincides with the reciprocal polar of the evolute. Now Halphen's formulæ, expressed in terms of k, l, show that the order of the singularity on the evolute is $k \sim l$, and if $k < l$ this singularity is at infinity, and the tangent meets it in l points; if $k > l$, it is at O, and the tangent meets it in k points. As to the first inverse, our results show that if $k < l$ the order of the singularity is k, and the tangent meets it in l points; while if $k > l$ these numbers are interchanged. Thus the order and class for the singularity on the evolute are in the two cases $(l-k,\ k)$, $(k-l,\ l)$; and for the singularity on the inverse they are $(k,\ l-k)$, $(l,\ k-l)$. Further, if $k<l$, OZ is the tangent at Z, and the reciprocal is at Ω; if $k>l$, IJ is the tangent at Z, and the reciprocal is at O; and if $k=l$ (a case in which a priori formulæ do not present themselves, and which Halphen therefore does not consider), the tangent is a line through Z inclined at a finite angle to OZ, ΩZ, and the reciprocal is somewhere on $O\Omega$, i. e. it is at a finite distance from O.

22. We have now to show that the reduction in class due to the compound singularity is the same as the reduction due to the constituent elements (nodes and cusps) enumerated by this process.

Let the total diminution due to the singularity of order k be D; when this singularity is inverted, we have a part of it preserved as a singularity on the inverse—let the effect of this on the class be D_1, so that $D-D_1$ is the effect of the central k-point in the original singularity; this central k-point is replaced by k points on the base, these giving $\varkappa_1$ proper contacts (improper contacts, that is cusps in the inverse singularity, are allowed for in D_1); we have to show that $D-D_1=k(k-1)+\varkappa_1$.

Take the conic of inversion a proper conic, choosing it so that I, J, as also OI, OJ, IJ, may have no specialty of position with regard to the given curve. Let the order and class of the given curve be m, n; and let the effect of all remaining singularities in diminishing the class be N, so that

$$n=m(m-1)-D-N. \tag{1}$$

The curve cuts OI, OJ, IJ in $m-k$, $m-k$, m, distinct points; we have therefore on the inverse, at I, an $(m-k)$-pt with distinct tangents; the same at J; and at O an m-pt with distinct tangents. On IJ we have also k points, and consequently

$$m'=2(m-k)+k,$$

$$\text{i. e.}\quad m'=2m-k; \tag{2}$$

hence

$$n'=m'(m'-1)-D_1-N-m(m-1)-2(m-k)(m-k-1), \tag{3}$$

for the ordinary proof, which is legitimate when the tangents are separate, shows that the diminution in class due to an m-pt is $m(m-1)$.

Now the n' tangents from I to the inverse are

(1) the inverses of the original n J-tangents;

(2) $m-k$ tangents at I, each counting twice;

(3) the line IJ, counting $\varkappa_1$ times;

$$\therefore\ n' = n + 2(m-k) + \varkappa_1. \tag{4}$$

Comparing (3) with (4), and making use of (1) and (2), we have

$$(2m-k)(2m-k-1) - D_1 - N - m(m-1) - 2(m-k)(m-k-1)$$
$$= m(m-1) + 2(m-k) + \varkappa_1 - D - N,$$

i. e.

$$D - D_1 = \varkappa_1 + 2(m-k) + 2(m-k)(m-k-1)$$
$$-(2m-k)(2m-k-1) + 2m(m-1),$$

i. e.

$$D - D_1 = k^2 - k + \varkappa_1 = k(k-1) + \varkappa_1. \tag{5}$$

Now the central k-point gives in all $k(k-1)/2$ dps, of which we have supposed $\varkappa_1$ to be cusps; we have therefore $\nu_1 = k(k-1)/2 - \varkappa_1$, and equation (5) becomes

$$D - D_1 = 2\nu_1 + 3\varkappa_1;$$

that is, the ν_1 nodes and $\varkappa_1$ cusps in composition have the same effect in reducing the class as when separate. We have therefore the equation

$$n = m(m-1) - 2\Sigma\nu - 3\Sigma\varkappa;$$

and from the reciprocal curve we have

$$m = n(n-1) - 2\Sigma\tau - 3\Sigma\iota;$$

where the $\Sigma\varkappa$, $\Sigma\iota$, refer to all consecutions, the $\Sigma\nu$, $\Sigma\tau$, to all remaining coincidences.

BRYN MAWR COLLEGE, BRYN MAWR, PA., *March*, 1893.